5. Problem sheet for Set Theory, Winter 2012

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Problem 15. (Finite set theory) Consider the axiom system HF (for *hereditarily finite*) consisting of EML, the foundation axiom, and the following induction principle

$$\forall x((x \in \mathcal{B} \wedge y \in \mathcal{B}) \rightarrow x \cup \{y\} \in \mathcal{B}) \rightarrow (\mathcal{B} = \emptyset \vee \mathcal{B} = V)$$

for every class term $\mathcal{B}$. Show that every axiom of ZF except Inf is provable in HF. Show that HF proves $\neg$Inf (notice that the proof of the recursion theorem does not use Inf).

Problem 16. (Ordinal arithmetic) Decide which of the following statements are true for all ordinals α, β, γ.

(a) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
(b) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
(c) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.
(d) $\alpha^{\beta+\gamma} = \alpha^{\beta} \cdot \alpha^{\gamma}$.
(e) $\alpha + \beta = \beta + \alpha$.
(f) $\alpha \cdot \beta = \beta \cdot \alpha$.

Problem 17. (Arithmetic) Prove

(a) $+[\omega \times \omega] := \{m + n \mid m \in \omega \wedge n \in \omega\} \subseteq \omega$.
(b) $\cdot[\omega \times \omega] := \{m \cdot n \mid m \in \omega \wedge n \in \omega\} \subseteq \omega$.
(c) Addition and multiplication on ω are commutative.
(d) Addition and multiplication on ω satisfy the usual monotonicity laws with respect to $<$.

Problem 18. (Rational numbers) Define addition and multiplication on $\mathbb{Q}$ and show distributivity.

There are 6 points for each problem. Please hand in your solutions on Monday, November 12 before the lecture.